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On modified versions of some solvable ordinary differential equations due to Chazy

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Abstract

We introduce modified versions, featuring a deformation parameter ω, of four explicitly solvable nonlinear autonomous ordinary differential equations (ODEs) introduced almost a century ago by Chazy (Chazy J 1909 *C. R. Acad. Sci., Paris* **148** 157–9). When the deformation parameter vanishes, $\omega = 0$, the modified ODEs reduce to those introduced by Chazy. When the deformation parameter ω does not vanish and is real (say, positive, $\omega > 0$), then all the nonsingular solutions of these modified ODEs, considered as functions of the real independent variable t (say, 'time'), are periodic with period $T = 2\pi/\omega$. For two of these modified ODEs, there is also a (small) subset of these solutions that are (also) periodic with a smaller period.

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1. Introduction

Almost a century ago Chazy [1] introduced the four (autonomous nonlinear) ordinary differential equations (ODEs)

$$w'' + w^3 w' = ww'(w^4 + 4w')^{1/2}, \tag{1.1}$$

$$w'' - 3w^5 = 3\mathrm{i}w^2(w'^2 - w^6)^{1/2}, \tag{1.2}$$

$$w'' - 2ww' = 2\mathrm{i}w'(w' - w^2 - c^2)^{1/2}, \tag{1.3}$$

$$w''' = 2(ww'' + w'^2) = (w^2)''. \tag{1.4}$$

Here $w \equiv w(\tau)$ is the dependent variable, τ is the independent variable, appended primes denote of course differentiations with respect to τ, and throughout this paper i denotes the imaginary unit, $\mathrm{i}^2 = -1$. These ODEs are remarkable for the following reasons.

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The first of these four ODEs, (1.1), possesses, in addition to the general solution

$$w(\tau) = A\tan(A^3\tau + B), \tag{1.5a}$$

the special solution

$$w(\tau) = (4/3)^{1/3}(\tau - \tau_b)^{-1/3}. \tag{1.5b}$$

Here A, B and τ_b are arbitrary constants. Note that the general solution (1.5*a*) can be obtained by dividing the ODE (1.1) by the square root it features in its right-hand side and by observing that both sides then become exact differentials, so that one can immediately integrate once; one then squares the resulting equation and again integrates easily, obtaining (1.5*a*). The special solution (1.5*b*) corresponds instead to the simultaneous vanishing of the square root in the right-hand side of (1.1), and of the left-hand side of this ODE. It should be emphasized that the special solution (1.5*b*) cannot be obtained as a limiting case of the general solution (1.2) and, in contrast to (1.2) which is meromorphic in the entire complex τ-plane (it clearly has simple poles at $\tau = \tau_n$, $\tau_n = A^{-3}[-B + (2n+1)\pi/2]$, $n =$ arbitrary integer), (1.5*b*) features a branch point of the order of one third at the (*a priori* arbitrary) value τ_b of the independent variable τ; hence this ODE, (1.1)—in contrast to what might be naively inferred from knowledge of its general solution (1.5*a*)—does not possess the 'Painlevé property' to only feature solutions the only *movable* singularities of which are poles. (Note that the movable singularities are those occurring at values of the independent variable that cannot be predicted *a priori*, namely at values that, in the context of the initial-value problem, do depend on the initial data; in the case of autonomous ODEs, all singularities are of this type.) Note moreover that the existence of the two different solutions (1.5*a*) and (1.5*b*) demonstrates the lack of uniqueness of the 'initial-value' problem for (1.1) whenever the initial data, say $w(0)$, $w'(0)$, satisfy the condition $w^4(0) + 4w'(0) = 0$, namely whenever they entail the vanishing of the square root in the right-hand side of (1.1).

Likewise, the second of these four ODEs, (1.2), possesses, in addition to the general solution

$$w(\tau) = A\wp(A^2\tau + B; 0, 4), \tag{1.6a}$$

the special solutions

$$w(\tau) = [\pm 2(\tau - \tau_b)]^{-1/2}. \tag{1.6b}$$

Here A, B and τ_b are again arbitrary constants, and $\wp(u; g_2, g_3)$ is the Weierstrass elliptic function. Clearly the same remarks made above apply here, with obvious adjustments; hence we do not repeat them.

The third of these four ODEs, (1.3), has been written above in a slightly more general form than used by Chazy, who wrote 1 in place of the constant c^2 [1]. Actually it is easily seen that, for $c \neq 0$, the 'cosmetic' rescaling

$$w(\tau) = c\tilde{w}(\tilde{\tau}), \qquad \tilde{\tau} = c\tau, \tag{1.7}$$

entails that $\tilde{w}(\tilde{\tau})$ satisfies an ODE analogous to (1.3) but with the constant c^2 replaced by unity, namely the Chazy version. Our motivation for using the more general form (1.3) is because below we shall also be interested in the $c = 0$ case.

As the two ODEs discussed above, (1.3) also features both a general solution and a special solution, which can both be written in explicit form. The former reads

$$w(\tau) = \exp(A\tau + B) + (A^2 - 4c^2)/(4A), \tag{1.8a}$$

and the latter reads

$$w(\tau) = -c\,\mathrm{cotan}[c(\tau - \tau_0)]. \tag{1.8b}$$

Here A, B and τ_0 are arbitrary constants. Some of the comments given above (after (1.5*b*)) are clearly also applicable to this case, hence they are not repeated. We note however one difference: the general solution is now entire (see (1.8*b*; in the previous two cases it was meromorphic, see (1.5*a*) and (1.6*a*)), and the special solution is now meromorphic (see (1.8*b*); in the two previous cases it had a branch point (see (1.5*b*) and (1.6*b*)). Also note that both the general solution (1.8*a*) and the special solution (1.8*b*) are also valid in the $c = 0$ case, when of course the ODE (1.3) reads

$$w'' - 2ww' = 2\mathrm{i}w'(w' - w^2)^{1/2}, \tag{1.9}$$

and its (general and special) solutions read

$$w(\tau) = \exp(A\tau + B) + A/4, \tag{1.10a}$$

$$w(\tau) = -(\tau - \tau_0)^{-1}. \tag{1.10b}$$

Note that the special solution (1.10*b*) is just a single pole, at $\tau = \tau_0$, with residue -1. (Incidentally one wonders why Chazy did not introduce the simpler ODE (1.9) rather than (1.3) with $c^2 = 1$, since (1.9) seems sufficient to illustrate his main point, namely that the analytic structure of the solutions of an ODE might be richer than one might infer by looking only at its general solution—of course, in the cases when the general solution does not include all the solutions of the ODE under consideration.)

Finally the fourth Chazy ODE, (1.4), which can obviously be integrated twice and then easily linearized (since after the first two integrations it becomes a first-order Riccati equation), admits the general meromorphic solution

$$\begin{aligned} w(\tau) = -[2(\tau - \tau_0)]^{-1} - (3A/2)(\tau - \tau_0)^{1/2}\{J'_{1/3}[A(\tau - \tau_0)^{3/2}] \\ + BJ'_{-1/3}[A(\tau - \tau_0)^{3/2}]\}\{J_{1/3}[A(\tau - \tau_0)^{3/2}] + BJ_{-1/3}[A(\tau - \tau_0)^{3/2}]\}^{-1}. \end{aligned} \tag{1.11}$$

Here A, B and τ_0 are three arbitrary constants, $J_{1/3}(u)$ and $J_{-1/3}(u)$ are the standard Bessel functions of the order of $1/3$ and $-1/3$ respectively, and the primes appended to these functions (see the numerator in the right-hand side) denote of course differentiation with respect to their arguments. Note that this solution, (1.11), is a meromorphic function of the independent variable τ: its poles occur at the zeros of the denominator in the right-hand side, while there is no singularity at $\tau = \tau_0$ (except in the special case $B = 0$).

After this terse survey of Chazy's results [1], let us come to the topic of this paper. Recently a 'trick' (amounting to a change of dependent and independent variables) has been introduced [2] which has the potential to yield, from certain classes of autonomous evolution equations, modified evolution equations that may also be autonomous and which feature many completely periodic solutions with *a priori* known periods. This approach has been applied to certain many-body problems, allowing us to evince much information on the phenomenology of their motions [2–8]. Its general applicability to both ODEs and partial differential equations (PDEs) has been surveyed [9]; classes of ODEs [10] and of PDEs [11], mainly of polynomial type, to which it is applicable have been identified; and its effects on two 'classical' equations due to Painlevé [12] and to Chazy [13] have been analyzed in [14] and in [10], respectively. In this paper we note that, remarkably, this trick is applicable—in the sense of generating, from autonomous ODEs, deformed ODEs which remain autonomous—to the three Chazy equations (1.1), (1.2) and (1.4), and as well to the ODE (1.3) in the $c = 0$ case, namely to (1.9). It can also be applied to (1.3) in the $c \neq 0$ case, but the deformed ODE thereby obtained is then not autonomous, see (1.14). In fact the four deformed ODEs obtained from (1.1)–(1.4) in this manner read

$$\ddot{z} - 5\mathrm{i}\omega\dot{z} - 4\omega^2 z = z(\dot{z} - \mathrm{i}\omega z)\{-z^2 + [z^4 + 4(\dot{z} - \mathrm{i}\omega z)]^{1/2}\}, \tag{1.12}$$

$$\ddot{z} - 4\mathrm{i}\omega\dot{z} - 3\omega^2 z = 3z^5 + 3\mathrm{i}z^2[(\dot{z} - \mathrm{i}\omega z)^2 - z^6]^{1/2}, \tag{1.13}$$

$$\ddot{z} - 3\mathrm{i}\omega\dot{z} - 2\omega^2 z = 2z(\dot{z} - \mathrm{i}\omega z) + 2\mathrm{i}(\dot{z} - \mathrm{i}\omega z)[\dot{z} - \mathrm{i}\omega z - z^2 - c^2 \exp(-2\mathrm{i}\omega t)]^{1/2}, \tag{1.14}$$

$$\dddot{z} - 6\mathrm{i}\omega\ddot{z} - 11\omega^2\dot{z} + 6\mathrm{i}\omega^3 z = 2(z\ddot{z} + \dot{z}^2) - 10\mathrm{i}\omega z\dot{z} - 6\omega^2 z^2. \tag{1.15}$$

Here $z \equiv z(t)$ is the (complex) dependent variable, t is the real independent variable (say, 'time'), and of course superposed dots denote differentiations with respect to t. Note that the three ODEs (1.12), (1.13) and (1.15) are indeed all autonomous, while the ODE (1.14) is autonomous only if $c = 0$. Each of these four complex ODEs, (1.12)–(1.15), could of course be rewritten as a system of two coupled real ODEs (by introducing the real and imaginary parts of the dependent variable, $z = x + \mathrm{i}y$, or its modulus and phase, $z = \rho\exp(\mathrm{i}\theta)$), but the resulting real 'equations of motion' are not sufficiently neat to deserve explicit display.

Clearly when the 'deformation parameter' ω vanishes, $\omega = 0$, these four ODEs, (1.12)–(1.15), reduce to the four Chazy ODEs (1.1)–(1.4) (up to trivial notational changes). In section 2 we derive (via the trick) these modified ODEs, (1.12)–(1.15), from the four Chazy ODEs (1.1)–(1.4), and we take advantage of the explicit solvability of the four Chazy ODEs—as described above—to exhibit the solutions of these four deformed ODEs, (1.12)–(1.15), and to thereby show that, if instead the deformation parameter ω does not vanish but it is real (without loss of generality, positive)

$$\omega > 0, \tag{1.16}$$

then all the nonsingular solutions (namely, those that do not blow up in a finite time) of these deformed ODEs, (1.12)–(1.15), are completely periodic with period T

$$T = 2\pi/\omega, \tag{1.17a}$$

$$z(t + T) = z(t). \tag{1.17b}$$

We also show that there is a (small) subset of special solutions of (1.12) that are moreover periodic with period $T/3$, and a (small) subset of special solutions of (1.13) that are moreover periodic with period $T/2$ (of course these solutions are also periodic with period T). Some final remarks are given in section 3.

2. The trick and its implications

The trick amounts to the following change of dependent and independent variables

$$z(t) = \exp(\mathrm{i}p\omega t)w(\tau), \tag{2.1a}$$

$$\tau = [\exp(\mathrm{i}q\omega t) - 1]/(\mathrm{i}q\omega), \tag{2.1b}$$

where p, q are two integers which shall be chosen appropriately in each case, see below. Note that, for $\omega = 0$, neither the dependent nor the independent variables change at all, and moreover that, for any value of the deformation parameter ω, (2.1b) entails that the origins of the (old and new) independent variables coincide, namely when the (real) time variable t vanishes, $t = 0$, the (complex) time-like variable τ also vanishes, $\tau = 0$. It is moreover plain that these formulae imply the following relations:

$$\dot{z}(t) - \mathrm{i}p\omega z(t) = \exp[\mathrm{i}(p + q)\omega t]w'(\tau), \tag{2.2a}$$

$$\ddot{z}(t) - \mathrm{i}(2p + q)\omega\dot{z}(t) - p(p + q)\omega^2 z = \exp[\mathrm{i}(p + 2q)\omega t]w''(\tau), \tag{2.2b}$$

$$\dddot{z} - 3\mathrm{i}(p + q)\omega\ddot{z} - (3p^2 + 6pq + 2q^2)\omega^2\dot{z} + \mathrm{i}p(p + q)(p + 2q)\omega^3 z = \exp[\mathrm{i}(p + 3q)\omega t]w'''. \tag{2.2c}$$

It is now easy to verify that the four modified ODEs (1.12)–(1.15) are obtained from the four Chazy ODEs (1.1)–(1.4) via the transformation (2.1), in each case with an appropriate assignment of the two integers p, q. In particular to go from (1.1) to (1.12) one chooses $p = 1, q = 3$, and this of course entails via (2.1) that the modified ODE (1.12) possesses the general solution (see (1.5*a*))

$$z(t) = \alpha \exp(\mathrm{i}\omega t) \tan \{[\alpha^3/(3\mathrm{i}\omega)] \exp(3\mathrm{i}\omega t) - \beta\}, \tag{2.3a}$$

as well as the special solution (see (1.5*b*))

$$z(t) = (4\mathrm{i}\omega)^{1/3}[1 - \gamma \exp(-3\mathrm{i}\omega t)]^{-1/3}. \tag{2.3b}$$

Here α, β and γ are arbitrary (complex) constants, the values of which become, of course, fixed in terms of the initial data in the context of the initial-value problem.

Clearly, a necessary and sufficient condition for the general solution (2.3*a*) to be nonsingular (for all real values of the independent variable t) is that the (generally complex) constants α and β satisfy the inequalities

$$|\alpha|^3 \neq 3\omega|\beta + (2n + 1)\pi/2|, \qquad n = 0, \pm 1, \pm 2, \ldots, \tag{2.4}$$

and all these nonsingular solutions are completely periodic with period T, see (1.17).

Likewise, the inequality

$$|\gamma| \neq 1 \tag{2.5}$$

is clearly necessary and sufficient to guarantee that the special solution (2.3*b*) be nonsingular for all real values of the independent variable t and that it be completely periodic with period T, see (1.17); moreover, if $|\gamma| < 1$ clearly (2.3*b*) is actually completely periodic with period $T/3$ (hence, of course, as well with period T, see (1.17)).

To go from (1.2) to (1.13) we instead set $p = 1, q = 2$, and we thereby conclude that the modified ODE (1.13) possesses the general solution (see (1.6*a*))

$$z(t) = \alpha \exp(\mathrm{i}\omega t) \wp([\alpha^2/(2\mathrm{i}\omega)] \exp(2\mathrm{i}\omega t) + \beta; 0, 4), \tag{2.6a}$$

as well as the special solutions (see (1.6*b*))

$$z(t) = [\gamma \exp(-2\mathrm{i}\omega t) \pm \mathrm{i}/\omega]^{-1/2}. \tag{2.6b}$$

Here α, β and γ are again arbitrary (complex) constants, the values of which become, of course, fixed in terms of the initial data in the context of the initial-value problem. It is clear from (2.6*a*) that the following conditions on the constants α and β are necessary and sufficient to guarantee that the general solution (2.6*a*) be nonsingular (as a function of the real variable t):

$$|\alpha|^2 \neq 2\omega|\beta - x_p + 2m\omega_1 + 2n\omega_2|, \qquad n, m = 0, \pm 1, \pm 2, \ldots. \tag{2.7}$$

Here of course ω_1, ω_2 denote the two semi-periods of the Weierstrass elliptic function $\wp(x; 0, 4)$ and x_p denotes the value at which this function has a pole, $\wp(x_p; 0, 4) = \infty$. Clearly all these nonsingular solutions, considered as functions of the real variable t, are completely periodic with period T, see (1.17). Likewise, the inequality

$$|\gamma| \neq \omega^{-1} \tag{2.8}$$

is clearly necessary and sufficient to guarantee that the special solution (2.6*b*) be nonsingular for all real values of the independent variable t and that it be completely periodic with period

T, see (1.17); moreover, if $|\gamma| < \omega^{-1}$ clearly (2.6b) is actually completely periodic with period $T/2$ (hence, of course, also with period T, see (1.17)).

The assignment appropriate to go from (1.3) to (1.14) is simply $p = q = 1$. Hence this ODE, (1.14), possesses the general solution (see (1.8a))

$$z(t) = \exp(\mathrm{i}\omega t)\{\beta \exp[\alpha \exp(\mathrm{i}\omega t)] - (\omega^2\alpha^2 + 4c^2)/(4\mathrm{i}\omega\alpha)\}, \tag{2.9a}$$

as well as the special solution (see (1.8b))

$$z(t) = c \exp(\mathrm{i}\omega t)\mathrm{cotan}\{c[(\mathrm{i}/\omega)\exp(\mathrm{i}\omega t) + \gamma]\}. \tag{2.9b}$$

Here α, β and γ are again arbitrary (complex) constants, the values of which become, of course, fixed in terms of the initial data in the context of the initial-value problem. It is plain that the general solution (2.9a) is nonsingular (as a function of the real variable t) for all values of α (of course $\alpha \neq 0$), of β and of c, and it is always completely periodic with period T (see (1.17)). As for the special solution (2.6b), clearly if $c \neq 0$ a necessary and sufficient condition to guarantee that it be nonsingular for all real values of the independent variable t is provided by the inequalities

$$|\gamma - 2\pi n/c| \neq \omega^{-1}, \qquad n = 0, \pm1, \pm2, \ldots, \tag{2.10}$$

while in the $c = 0$ case (when the ODE (1.14) becomes autonomous) the special solution takes the simpler form

$$z(t) = [(\mathrm{i}/\omega) + \gamma \exp(-\mathrm{i}\omega t)]^{-1}, \tag{2.11}$$

and the necessary and sufficient condition to guarantee that it be nonsingular for all real values of t is again provided by the inequality (2.8). It is of course plain that these special solutions, see (2.9b) (if $c \neq 0$) or (2.11) (if $c = 0$), are also completely periodic with period T, see (1.17), whenever they are nonsingular.

The same assignment, $p = q = 1$, is the appropriate one to go from (1.4) to (1.15). Hence this ODE, (1.15), possesses the general solution (see (1.11))

$$\begin{aligned} z(t) = \exp(\mathrm{i}\omega t)\{ &- (2[\beta - (\mathrm{i}/\omega)\exp(\mathrm{i}\omega t)])^{-1} \\ &- (3A/2)[\beta - (\mathrm{i}/\omega)\exp(\mathrm{i}\omega t)]^{1/2}\{J'_{1/3}(A[\beta - (\mathrm{i}/\omega)\exp(\mathrm{i}\omega t)]^{3/2}) \\ &+ BJ'_{-1/3}(A[\beta - (\mathrm{i}/\omega)\exp(\mathrm{i}\omega t)]^{3/2})\}\{J_{1/3}(A[\beta - (\mathrm{i}/\omega)\exp(\mathrm{i}\omega t)]^{3/2}) \\ &+ BJ_{-1/3}(A[\beta - (\mathrm{i}/\omega)\exp(\mathrm{i}\omega t)]^{3/2})\}^{-1}\}. \end{aligned} \tag{2.12}$$

Here β, A and B are again arbitrary (complex) constants, the values of which become, of course, fixed in terms of the initial data in the context of the initial-value problem. It is plain that this general solution (2.12)—which now includes all solutions of the deformed ODE (1.15)—is completely periodic with period T, see (1.17), unless it is singular, and that a necessary and sufficient condition to guarantee that it be singular (namely, that it blow up at a finite real value of the time t), is the validity of the following condition (on the three integration constants β, A, B)

$$|\beta - (x_n/A)^{2/3}| = |\omega|^{-1}, \tag{2.13a}$$

where $x_n \equiv x_n(B)$ is one of the zeros of the function $J_{1/3}(x) + BJ_{-1/3}(x)$

$$J_{1/3}(x_n) + BJ_{-1/3}(x_n) = 0. \tag{2.13b}$$

3. Final remarks and outlook

The fact that a nonlinear evolution ODE possesses many periodic solutions, all of them with the same period (as exemplified by the findings reported above), may appear surprising because, in the context of the initial-value problem for nonlinear evolution equations, a generic small change of the initial data that yield a completely periodic solution is expected to produce either a non completely periodic (possibly multiply periodic) solution or another completely periodic solution but with a different period. This paradigm is however negated by the possibility to perform a change of independent variables, such that the old variable itself becomes a periodic function of the new variable, with a given period. It then becomes natural to expect that the new evolution equation obtained in this manner possesses many periodic solutions, all of them with that same period. This is the essence of the 'trick' introduced recently [2] and exploited in this paper and in several other papers [3–11, 14], the contents of which have been tersely outlined in the introductory section 1. What is perhaps less trivial is the possibility to relate via this simple trick autonomous ODEs to autonomous ODEs. Classes of ODEs for which this is possible can be identified [10], but it is also of interest to look at specific examples, particularly when these have a historical significance, and they moreover allow, due to their explicit solvability, a more complete treatment than is otherwise possible. This has provided the main motivation to treat the cases reported above.

The following remark provided an additional, more specific, motivation to focus on the ODEs considered above and in other recent papers [10, 14]. The trick (2.1) has the interesting feature to translate the analyticity properties in the complex variable τ of the solutions of a certain ODE into properties of periodicity in the real variable t of another ('deformed') ODE. In particular, it is for instance clear that the application to a certain ODE of the trick, see (2.1), produces a deformed ODE that possesses many completely periodic solutions with period T, see (1.17) (as functions of the real time t), only if the solutions of the original ODE have a simple analytic structure—say, if they are meromorphic functions of the complex time-like variable τ (see (2.1)). It is therefore natural to pay special attention to ODEs introduced in 'historical' papers [1, 12, 13] focused on the classification of ODEs on the basis of the analyticity properties of their solutions. In this context it is remarkable that many of these historical ODEs appear particularly suited to the application of the trick (2.1), in the sense of transforming autonomous ODEs into autonomous ODEs—as demonstrated above and elsewhere [10, 14].

But the most promising applications of the 'trick' are probably in the applicative context of modelling cyclic phenomena [9], and of studying many-body problems that exhibit interesting behaviours [2–8].

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